CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Differential Equations 215 (2005) 429–447

Journal of Differential Equations

www.elsevier.com/locate/jde

Local existence and Gevrey regularity of 3-D Navier–Stokes equations with ℓ_p initial data

Animikh Biswas*, [1]

Department of Mathematics & Statistics, University of North Carolina at Charlotte, 9201 University City Blvd., Charlotte NC-28223, USA

Received 25 June 2004; revised 3 December 2004
Available online 26 February 2005

Abstract

We obtain local existence and Gevrey regularity of 3-D periodic Navier–Stokes equations in case the sequence of Fourier coefficients of the initial data is in ℓ_p $(p<3/2)$. The ℓ_p norm of the sequence of Fourier coefficients of the solution and its analogous Gevrey norm remains bounded on a time interval whose length depends only on the size of the body force and the ℓ_p norm of the Fourier coefficient sequence of the initial data. The control on the Gevrey norm produces explicit estimates on the analyticity radius of the solution as in Foias and Temam (J. Funct. Anal. 87 (1989) 359–369). The results provide an alternate approach in estimating the space-analyticity radius of solutions to Navier–Stokes equations than the one presented by Grujić and Kukavica (J. Funct. Anal. 152 (1998) 447–466).
© 2005 Elsevier Inc. All rights reserved.

MSC: 35Q30; 35Q35; 76D05

Keywords: Navier–Stokes equations; Gevrey regularity

1. Introduction

In this article, we consider the Navier–Stokes equations (NSE) with space-periodic boundary condition. A method for estimating the space-analyticity radius of solutions of NSE in this setup was introduced by Foias and Temam in [FT]. The basic idea

* Fax: +1 704 687 6415.
E-mail address: abiswas@uncc.edu.
[1] Supported in part by NSF Grant number DMS–0074460 and Junior Summer Research Fellowship (2003) at UNC, Charlotte.

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2004.12.012

in [FT] of interpolating between a suitably defined analyticity (Gevrey) norm and a Sobolev norm leads to a very simple energy method, eliminating the need of traditional estimates on higher order derivatives (for example, as in [M]). This also provides an explicit estimate of the radius of analyticity in terms of the Sobolev H^1-norm of the initial data and the size of the forcing term. Subsequently, in [GK], Grujić and Kukavica provided an estimate of the analyticity radius, where they assumed the initial data to be in L^q $(q > 3)$. Unlike in [FT], instead of estimating the Gevrey norm directly, Grujić and Kukavica achieve the relevant estimates on the analyticity radius by interpolating between the L^q norm of the initial data and the L^q norm of the complexified solution.

As mentioned before, we consider the 3-D NSE with space-periodic boundary condition. In this setup, the NSE can be reformulated in terms of its Fourier coefficients. The resulting system can be regarded as a nonlinear evolution equation in an appropriate sequence space. This is the so-called wavevectors formulation of the NSE (see [F] for a detailed exposition). We assume that the initial data is such that the ℓ_p $(p < \frac{3}{2})$ norm of its sequence of Fourier coefficients (which henceforth will be referred to as the ℓ_p norm of the periodic function) is finite. By employing only elementary Functional Analytic techniques which completely bypasses the Sobolev inequalities, we prove that there exists a local in time solution of the 3D-NSE with bounded ℓ_p and Gevrey norms. It should be noted here that the Hausdorff–Young inequality states that for $1 \leqslant p \leqslant 2$, the ℓ_p norm of a periodic function dominates its L^q norm, where q is the Hölder conjugate of p. Thus our assumption on the initial data is stronger than that in [GK] in case of estimate of analyticity radius, and that of [FK] and [GM] in case of local existence results. However, since we control the ℓ_p norm of the solution, our conclusion is also similarly strengthened. Consequently, the results in [FK,GK] or [GM] do not directly imply the results presented here. Moreover, our results provide generalization of those obtained using energy methods by Foias (see [F]), where it was assumed that the initial data is in ℓ_1. The proof given here employs fixed point methods and is motivated by [FK,K,GM].

The paper is organized as follows. In Section 2, we establish notation and discuss some preliminary material. In Section 3, we first obtain a local in time solution which is bounded in ℓ_p $(p < \frac{3}{2})$ norm, in case the initial data also belongs to ℓ_p $(p < \frac{3}{2})$. Subsequently, we show that this solution is regular. Finally, in Section 3, we obtain local in time solution, which is bounded in the Gevrey norm. This provides an alternative approach to that of [GK] and is more in the spirit of the results in [FT]. The treatment here is essentially self-contained and elementary.

2. Notation and preliminaries

We consider the Navier–Stokes equations of viscous incompressible fluids in $\Omega = [0, L]^3$ with space periodic boundary condition:

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2.2}$$

The unknown (real-valued) functions are the vector-valued velocity function $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the scalar-valued pressure $p = p(x,t), x \in \mathbb{R}^3, t \geqslant 0$. The volume force $\mathbf{f}(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$ is given and $v > 0$ is the kinematic viscosity. For notational simplicity, henceforth, we will set $v \equiv 1$. We assume that $\mathbf{f}, \mathbf{u}, p$ are periodic in space variables with period L. For a L-periodic complex-valued scalar or vector function ϕ which is integrable over Ω, we define its Fourier coefficients by

$$\hat{\phi}(k) = \frac{1}{L^3}\int_\Omega e^{-\frac{2\pi\imath}{L}k\cdot x}\phi(x)\,dx, \quad (k \in \mathbb{Z}^3),$$

and its corresponding Fourier series is defined by

$$\sum_{k\in\mathbb{Z}^3} \hat{\phi}(k)e^{-\frac{2\pi\imath}{L}k\cdot x}.$$

If ϕ, ψ are two complex vector functions, square-integrable on Ω, Parseval's identity says that

$$\frac{1}{L^3}\int_\Omega \phi(x)\cdot\psi(x)^*\,dx = \sum_{k\in\mathbb{Z}^3} \phi(k)\cdot\psi(k)^*,$$

where for $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$

$$b^* = (\bar{b}_1, \bar{b}_2, \bar{b}_3), \quad \mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Also, for scalar- or vector-valued function $\phi = \phi(x,t) : \mathbb{R}^3\times[0,T] \to \mathbb{C}^n,\ T \leqslant \infty,\ n \in \mathbb{N}$, L-periodic in the space variable, we denote by $(\phi(k,t))_{k\in\mathbb{Z}^3}$ the sequence of Fourier coefficients of the function $\phi(\cdot,t)$.

Rewriting (2.1) and (2.2) in terms of its Fourier coefficient as is done in [F], one obtains the so-called wavevectors formulation of the NSE as follows:

$$\frac{d}{dt}\mathbf{u}(k,t) = \mathbf{f}(k,t) - \frac{2\pi\imath}{L}kp(k,t) - \left(\frac{2\pi}{L}\right)^2 |k|^2\vec{u}(k,t) - Q[\mathbf{u},\mathbf{u}](k,t), \tag{2.3}$$

$$k\cdot\mathbf{u}(k,t) = 0 \quad (k \in \mathbb{Z}^3), \tag{2.4}$$

where, for two $\mathbb{C}^3$ or $\mathbb{R}^3$-valued sequences $(\mathbf{u}(k))_{k\in\mathbb{Z}^3}, (\mathbf{v}(k))_{k\in\mathbb{Z}^3}$,

$$Q[\mathbf{u},\mathbf{v}](k) = \frac{2\pi\imath}{L}\sum_{h\in\mathbb{Z}^3}(k\cdot\mathbf{u}(k))\mathbf{v}(k-h).$$

Note that if the sequences $(\mathbf{u}(k))_{k\in\mathbb{Z}^3}$ and $(\mathbf{v}(k))_{k\in\mathbb{Z}^3}$ are square-summable, $Q[\mathbf{u},\mathbf{v}](k)$ is well defined for each $k\in\mathbb{Z}^3$. Since the functions $\mathbf{u},\mathbf{f},p$ are all real, we also have

$$\mathbf{u}(-k,t)=\mathbf{u}(k,t)^*,\quad p(-k,t)=\bar{p}(k,t),\quad \mathbf{f}(-k,t)=\mathbf{f}(k,t)^*,\qquad (k\in\mathbb{Z}^3, t\geqslant 0).$$

Moreover, without loss of generality (see [T]), we may also assume

$$\mathbf{u}(0,t)=\mathbf{f}(0,t)=\mathbf{0},\qquad p(0,t)=0,\qquad (k\in\mathbb{Z}^3).$$

Using (2.4) and taking dot-product with k on both sides of (2.3), one readily obtains

$$\frac{2\pi\imath}{L}\,p(k,t)=[k\cdot\mathbf{f}(k,t)-k\cdot Q[\mathbf{u},\mathbf{u}](k,t)]/|k|^2.$$

Reintroducing this in (2.3) we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{u}(k,t)&=\mathbf{g}(k,t)-\left(\frac{2\pi}{L}\right)^2|k|^2\mathbf{u}(k,t)-B[\mathbf{u},\mathbf{u}](k,t),\\ k\cdot\mathbf{u}(k,t)&=0\quad (k\in\mathbb{Z}^3),\end{aligned}$$

where

$$B[\mathbf{u},\mathbf{v}](k,t)=Q[\mathbf{u},\mathbf{v}](k,t)-\frac{k(k\cdot Q[\mathbf{u},\mathbf{v}](k,t))}{|k|^2},\quad \mathbf{g}(k,t)=\mathbf{f}(k,t)-\frac{k(k\cdot\mathbf{f}(k,t))}{|k|^2}.$$

In view of the above discussion, following the treatment in [F], one may thus obtain an infinite dimensional ODE formulation of Navier–Stokes equations in sequence space. We describe the set-up below in detail.

Let

$$\mathcal{K}=\{\vec{v}=\{(\vec{v}(k))_{k\in\mathbb{Z}^3}:\vec{v}(k)\in\mathbb{C}^3,\ \vec{v}(0)=0,\vec{v}(-k)=\vec{v}(k)^*,\ k\cdot\vec{v}(k)=0\}.$$

The vector space $\mathcal{K}$, endowed with the topology of coordinate wise convergence, is a Frechet space with respect to the metric

$$d(\vec{v}_1,\vec{v}_2)=\sum_{k\in\mathbb{Z}^3}\frac{|\vec{v}_1(k)-\vec{v}_2(k)|}{1+|\vec{v}_1(k)-\vec{v}_2(k)|}\,2^{-|k|^2}.$$

For $\alpha \geqslant 0$ and $p \geqslant 1$ define

$$V_{\alpha,p} = \left\{ \vec{u} \in \mathcal{K} : \|\vec{u}\|_{\alpha,p} := \left(\sum_{k \in \mathbb{Z}^3} [|k|^{\alpha} |u(k)|]^p \right)^{1/p} < \infty \right\}.$$

Clearly, $V_{\alpha_1,p} \subset V_{\alpha_2,p}$ if $\alpha_1 \geqslant \alpha_2$ and in this case,

$$\|\vec{u}\|_{\alpha_2,p} \leqslant \|\vec{u}\|_{\alpha_1,p} \quad (\vec{u} \in V_{\alpha_1,p}). \tag{2.5}$$

In case $\alpha = 0$, for notational simplicity, we will refer to $V_{0,p}$ as V_p and the corresponding norm $\|\cdot\|_{0,p}$ is denoted by $\|\cdot\|_p$. For $\vec{u}, \vec{v} \in V_p$, define $Q[\vec{u}, \vec{v}]$ and $B[\vec{u}, \vec{v}]$ by

$$Q[\vec{u}, \vec{v}](k) = \frac{2\pi\imath}{L} \sum_{h \in \mathbb{Z}^3} (k \cdot u(k)) v(k-h),$$

$$B[\vec{u}, \vec{v}](k) = Q[\vec{u}, \vec{v}](k) - \frac{k \cdot Q[\vec{u}, \vec{v}](k)}{|k|^2} k. \tag{2.6}$$

We will first state here Young's inequality for convolution. For $\vec{u}, \vec{v} \in \ell_p(\mathbb{Z}^3)$ and $p \leqslant 2$, Young's inequality implies that the convolution $\vec{w} = \vec{u} * \vec{v}$

$$w(k) = \sum_{h \in \mathbb{Z}^3} u(h) v(k-h), \quad \vec{w} \in \ell_r(\mathbb{Z}^3), \ \ r = \frac{p}{2-p}, \ \ \|\vec{w}\|_r \leqslant \|\vec{u}\|_p \|\vec{v}\|_p. \tag{2.7}$$

Note that by Young's inequality, for all $\vec{u}, \vec{v} \in V_p$, $Q[\vec{u}, \vec{v}](k)$ is well-defined for each $k \in \mathbb{Z}^3$. Moreover, it can be easily checked that

$$|B[\vec{u}, \vec{v}](k)| \leqslant |Q[\vec{u}, \vec{v}](k)| \quad (k \in \mathbb{Z}^3) \tag{2.8}$$

and $B[\vec{u}, \vec{v}]$ is in $\mathcal{K}$.

Let A be the positive, unbounded, densely defined operator on V_p given by

$$A\vec{u} = \left(\left(\frac{2\pi}{L} \right)^2 |k|^2 u(k) \right)_{k \in \mathbb{Z}^3}, \quad (\vec{u} \in V_p).$$

We note here that for any $\vec{v} \in V_{\alpha+2\delta,p}$ for some $\alpha \geqslant 0$ and $\delta \geqslant 0$

$$\|A^{\delta} \vec{v}\|_{\alpha,p} = \left(\frac{2\pi}{L} \right)^{2\delta} \|\vec{v}\|_{\alpha+2\delta,p}. \tag{2.9}$$

For $T < \infty$ denote

$$L^1([0,T];V_{\alpha,p}) = \left\{ \vec{v}(\cdot) : [0,T] \to V_{\alpha,p} : \ \vec{v}(\cdot) \text{ measurable}, \ \int_0^T \|\vec{v}(s)\|_{\alpha,p}\, ds < \infty \right\}$$

and

$$L^\infty([0,T];V_{\alpha,p}) = \left\{ \vec{v}(\cdot) : [0,T] \to V_{\alpha,p} : \ \vec{v}(\cdot) \text{ measurable}, \ \sup_{0\leqslant s\leqslant T} \|\vec{v}(s)\|_{\alpha,p} < \infty \right\}.$$

For a function $\vec{v}(\cdot) : [0,T] \to V_{\alpha,p}$, for each $k \in \mathbb{Z}^3$, we denote by $\vec{v}(k,t)$ the kth "coordinate" of $\vec{v}(t)$. Moreover, $C([0,T];X)$, $X = V_{\alpha,p}$ or $\mathcal{K}$, denotes the set of all X-valued continuous function on $[0,T]$, where the continuity is with respect to the norm topology if $X = V_{\alpha,p}$ and, with respect to the "coordinate wise convergence" defined before, if $X = \mathcal{K}$.

Definition 1. Given $\vec{u}_0 \in V_p$, $p \leqslant 2$ and $\vec{g}(\cdot) \in L^1([0,T];V_p)$, a measurable function $\vec{u}(\cdot)$ is said to be a weak solution of the Navier–Stokes initial value problem if it is in $C([0,T];K) \cap L^\infty([0,T];V_p)$ and satisfies

$$\begin{aligned} \frac{d}{dt}\vec{u}(k,t) &= \vec{g}(k,t) - \left(\frac{2\pi}{L}\right)^2 |k|^2 \vec{u}(k,t) - B[\vec{u},\vec{u}](k,t), \\ \vec{u}(k,0) &= \vec{u}_0(k), \quad (t > 0, k \in \mathbb{Z}^3). \end{aligned} \tag{2.10}$$

For $\vec{u}(\cdot) \in C([0,T];V_p)$, we denote the sup norm

$$\|\vec{u}(\cdot)\| = \sup_{0\leqslant t\leqslant T} \|\vec{u}(t)\|_p.$$

The space $C([0,T];V_p)$, equipped with the sup norm, is a closed subspace of $L^\infty([0,T];V_p)$ and is a Banach space.

Definition 2. Given $\vec{u}_0 \in V_p$, $p \leqslant 2$ and $\vec{g}(\cdot) \in L^1([0,T];V_p)$, a function $\vec{u}(\cdot)$ is said to be a mild solution of the NSE if it is in $C([0,T];V_p)$ and satisfies the integral equation

$$\vec{u}(t) = e^{-tA}\vec{u}_0 + \int_0^t e^{-(t-s)A}\vec{g}(s)\, ds - \int_0^t e^{-(t-s)A} B[\vec{u}(s),\vec{u}(s)]\, ds. \tag{2.11}$$

Definition 3. We say that $\vec{u}$ is a Leray-strong solution of the NSE if it is a weak solution which moreover satisfies

$$\vec{u}(\cdot) \in L^\infty([0,T];V_{1,p}). \tag{2.12}$$

Remark 1. From our results in the next section, it will follow that if $\vec{u}_0 \in V_p,\ 1 \leqslant p < \frac{3}{2}$ and for adequate $\vec{g}(\cdot)$, there exists a weak solution as in Definition 1 which moreover satisfies $\sup_{t\in[\varepsilon,T]} \|\vec{u}(t)\|_{1,2} < \infty$ for any $\varepsilon > 0$. From [T], it then follows that the function $\mathbf{u}(\mathbf{x},t) := \sum_{k\in\mathbb{Z}^3} \vec{u}(\mathbf{k},t)e^{\frac{2\pi\imath}{L}\mathbf{k}\cdot\mathbf{x}}$ belongs to V for $t \in [\varepsilon, T]$ and is thus a smooth classical solution of the Navier–Stokes equation in that interval. Here, following [T], we denote by V the set of all periodic, divergence free functions on $\Omega = [0, L]^3$ which are in the Sobolev space $\mathbb{H}^1$.

3. Local existence and regularity

Let $\vec{g}(\cdot) \in L^1([0, T']; V_p)$ where $0 < T' < \infty$. For $\vec{u}_0 \in V_p$ define

$$\vec{G}(t) = e^{-tA}\vec{u}_0 + \int_0^t e^{-(t-s)A}\vec{g}(s)\,ds, \quad 0 \leqslant t \leqslant T'. \tag{3.1}$$

Remark 2. Since e^{-tA} is a contraction semigroup on V_p we have

$$\|\vec{G}(t)\| \leqslant M := \|\vec{u}_0\|_p + \int_0^{T'} \|\vec{g}(s)\|_p, \quad (t \geqslant 0). \tag{3.2}$$

The assumption $\vec{g}(\cdot) \in L^1([0, T']; V_p)$ and the fact that e^{-tA} is a contractive semigroup on V_p implies that $\vec{G}(\cdot)$ belongs to $C([0, T']; V_p)$.

The main results of this section, stated below, give existence and regularity of solutions to Navier–Stokes equations in the space $L^\infty([0, T]; V_p)$, $1 \leqslant p < 3/2$, where the existence time T depends on the initial data (assumed to be in V_p), and the size of the body force.

Theorem 1. *Assume that* $1 \leqslant p < \frac{3}{2}$ *and* M *be as in* (3.2). *Let* $\delta_p = \delta > 0$ *if* $1 < p < \frac{3}{2}$ *and* $\delta_1 = \delta = 0$ *for* $p = 1$. *Then, for an adequate constant* $C = C(p, \delta)$ *and* $T < \min\{T', \frac{C}{M^{2p/(3-2p)+\delta}}\}$, *there exists* $\vec{u}(\cdot)$ *in* $C([0, T]; V_p)$ *with* $\sup_{0\leqslant t\leqslant T} \|\vec{u}(t)\| < 2M$ *which satisfies* (2.10) *and* (2.11). *For* $1 < p < \frac{3}{2}$, *the constant* C *may depend on* p *and* δ *only as* $p \to \frac{3}{2}$ *and* $\delta \to 0$.

The next theorem concerns the regularity of the solutions obtained in Theorem 1.

Theorem 2. *Assume that the force* $\vec{g}(\cdot)$ *satisfies*

$$\sup_{0\leqslant t\leqslant T} \|\vec{g}(t)\|_p < \infty \tag{3.3}$$

and the initial data u_0 *is in* V_p, $1 \leqslant p < 3/2$. *The mild* (*as well as weak*) *solution* $\vec{u}(t)$ *obtained in Theorem* 1 *is in fact a strong solution on any interval* $[\varepsilon, T]$, $\varepsilon > 0$, *which moreover satisfies*

$$\sup_{0 \leqslant t \leqslant T} t^{\alpha/2} \|\vec{u}(t)\|_{\alpha,p} < \infty \tag{3.4}$$

for all $0 \leqslant \alpha \leqslant 1$.

In order to prove the theorems stated above, we will need some more notation. Define

$$E = \{\vec{v}(\cdot) \in C([0,T]; V_p) : \|\vec{v}(\cdot) - \vec{G}(\cdot)\| = \sup_{0 \leqslant t \leqslant T} \|v(t) - \vec{G}(t)\|_p \leqslant M\}. \tag{3.5}$$

Note that

$$\|\vec{v}(\cdot)\| \leqslant 2M \quad \text{for all } \vec{v}(\cdot) \in E. \tag{3.6}$$

Let $S : E \to C([0,T]; V_p)$ be the map defined by the formula

$$(S\vec{v})(t) = \vec{G}(t) - \int_0^t e^{-(t-s)A} B[\vec{v}(s), \vec{v}(s)]\, ds, \quad (\vec{v} \in E). \tag{3.7}$$

We will show below that S maps E to E. We first state an elementary inequality which will be used repeatedly. For $a > 0$, $b > 0$, we have

$$f(\lambda) = \lambda^a e^{-b\lambda} \leqslant \left(\frac{a}{e}\right)^a \frac{1}{b^a} \quad \text{for all } \lambda > 0. \tag{3.8}$$

Lemma 1. *Assume that* $\vec{u}, \vec{v} \in V_{\alpha,p}$ *for some* $\alpha \geqslant 0$ *and let* $\eta > 0$. *For* $1 < p < 2$, *let* $\beta > \frac{3(p-1)}{p}$; *and for* $p = 1$, *let* $\beta = 0$. *Then, there exists a constant* $C = C(p, \alpha, \beta)$ *such that*

$$\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{\alpha,p} \leqslant C \frac{1}{\eta^{\frac{(\beta+1)}{2}}} \|\vec{u}\|_{\alpha,p} \|\vec{v}\|_{\alpha,p} \tag{3.9}$$

Proof. Recalling (2.8), we have

$$\begin{aligned} &\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{\alpha,p}^p \\ &\quad = \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2 |k|^2} |B[\vec{u}, \vec{v}](k)|^p \leqslant \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2 |k|^2} |Q[\vec{u}, \vec{v}](k)|^p \end{aligned}$$

$$= \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h (k \cdot u(h)) v(k-h) \right|^p$$

$$\leqslant \left(\frac{2\pi}{L}\right)^2 \sum_k |k|^{\alpha p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h |k||u(h)||v(k-h)| \right|^p$$

$$= \left(\frac{2\pi}{L}\right)^2 \sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| |k|^\alpha \sum_h |u(h)||v(k-h)| \right|^p . \tag{3.10}$$

Continuing from (3.10), we obtain

$$\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{\alpha,p}^p$$

$$\leqslant C \sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h (|h|^\alpha + |k-h|^\alpha) |u(h)||v(k-h)| \right|^p \tag{3.11}$$

$$\leqslant C \left(\sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h |h|^\alpha |u(h)||v(k-h)| \right|^p \right.$$

$$\left. + \sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h |u(h)||k-h|^\alpha |v(k-h)| \right|^p \right). \tag{3.12}$$

To obtain (3.11) and (3.12) above, we used the inequality $(a+b)^\alpha \leqslant C(a^\alpha + b^\alpha),\ a, b \geqslant 0$, where the constant C may depend only on α.

Let $w_1(k) = \sum_h |h|^\alpha |u(h)||v(k-h)|$ and $\vec{w}_1 = (w_1(k))_{k \in \mathbb{Z}^3}$. By Young's inequality for convolution, we have

$$\|\vec{w}_1\|_r \leqslant \|\vec{u}\|_{\alpha,p} \|\vec{v}\|_p \leqslant \|\vec{u}\|_{\alpha,p} \|\vec{v}\|_{\alpha,p}, \quad r = \frac{p}{2-p}. \tag{3.13}$$

Now suppose $1 < p < 2$ and $\beta > \frac{3(p-1)}{p}$. The first term in the inequality in (3.12) is

$$\sum_k |k|^{p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h |h|^\alpha |u(h)||v(k-h)| \right|^p$$

$$= \sum_k |k|^p |k|^{\beta p} e^{-p\eta(\frac{2\pi}{L})^2|k|^2} \frac{|w_1(k)|^p}{|k|^{\beta p}} \leqslant \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \sum_k \frac{|w_1(k)|^p}{|k|^{\beta p}}$$

$$\leqslant \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \|\vec{w}_1\|_r^p \left(\sum_k \frac{1}{|k|^{\frac{\beta p r}{r-p}}} \right)^{\frac{r-p}{r}} \leqslant \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \|\vec{u}\|_{\alpha,p}^p \|\vec{v}\|_{\alpha,p}^p, \tag{3.14}$$

where to obtain the inequality in the second last line above we used (3.8), and to obtain the first inequality in the last line above we applied Holder's inequality. To obtain the last inequality we used (3.13) and the fact that $\sum_k \frac{1}{|k|^{\frac{\beta pr}{r-p}}} < \infty$ if $\frac{\beta pr}{r-p} > 3$. This is indeed the case since $r = \frac{p}{2-p}$ and $\beta > \frac{3(p-1)}{p}$.

When $p = 1$ we obtain the same inequality as in (3.14) with $\beta = 0$. To see this, note that the first term in the inequality in (3.12) gives

$$\sum_k |k| e^{-\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h |h|^\alpha |u(h)||v(k-h)| \right| \leqslant \frac{C}{\eta^{\frac{1}{2}}} \sum_k |w_1(k)| \leqslant \frac{C}{\eta^{\frac{1}{2}}} \|\vec{u}\|_{\alpha,1} \|\vec{v}\|_{\alpha,1},$$

where to obtain the first inequality above, we used (3.8) and to obtain the last inequality, we used (3.13).

A similar computation shows that for the second term in (3.12) we have

$$\sum_k |k| e^{-\eta(\frac{2\pi}{L})^2|k|^2} \left| \sum_h |u(h)||k-h|^\alpha |v(k-h)| \right| \leqslant \frac{C}{\eta^{\frac{1}{2}}} \|\vec{u}\|_{\alpha,1} \|\vec{v}\|_{\alpha,1}. \tag{3.15}$$

Putting together inequalities (3.12), (3.14) and (3.15), the proof of the lemma is now complete. □

We will now prove that the map S defined in (3.7) takes E into $C([0,T];V_p)$. Note first that if $p < 3/2$, then there exists $\frac{3(p-1)}{p} < \beta < 1$.

Lemma 2. *Assume that* $1 \leqslant p < \frac{3}{2}$ *and let* $\beta = 0$ *for* $p = 1$; *and* $\frac{3(p-1)}{p} < \beta < 1$ *for* $1 < p < \frac{3}{2}$. *For* $\vec{v} \in E$, *we have* $S\vec{v}$ *is in* $C([0,T];V_p)$ *and a constant* $C = C(p,\beta)$ *such that*

$$\|(S\vec{v} - G)(t)\|_p \leqslant C M^2 T^{(1-\beta)/2}. \tag{3.16}$$

Proof. For $\vec{v} \in E$ and $t < T$ using Lemma 1 (with $\alpha = 0$) we have

$$\begin{aligned} \|(S\vec{v} - G)(t)\|_p &= \left\| \int_0^t e^{-(t-s)A} B[\vec{v}(s), \vec{v}(s)]\, ds \right\|_p \\ &\leqslant \int_0^t \|e^{-(t-s)A} B[\vec{v}(s), \vec{v}(s)]\|_p\, ds \\ &\leqslant C \int_0^t \frac{1}{(t-s)^{(\beta+1)/2}} \|\vec{v}(s)\|_p^2\, ds \\ &\leqslant \frac{8C}{1-\beta} M^2 T^{(1-\beta)/2}, \end{aligned}$$

where, to obtain the inequalities in the last line, we used (3.9) and (3.6). This establishes that $S\vec{v}$ is in $L^\infty([0,T]; V_p)$. In fact, the above calculations also show that $S\vec{v}$ belongs to $C([0,T]; V_p)$. $\square$

Lemma 3. *Assume that* $1 \leqslant p < \frac{3}{2}$ *and let* $\beta = 0$ *for* $p = 1$; *and* $\frac{3(p-1)}{p} < \beta < 1$ for $1 < p < \frac{3}{2}$. *For* $\vec{v}, \vec{w} \in E$, *and adequate* $C = C(p, \beta)$, *we have*

$$\|(S\vec{v} - S\vec{w})(t)\|_p \leqslant CMT^{(1-\beta)/2} \sup_{0 \leqslant t \leqslant T} \|(\vec{v} - \vec{w})(t)\|_p. \tag{3.17}$$

Proof. Note that by linearity,

$$B[\vec{v}, \vec{v}] - B[\vec{w}, \vec{w}] = B[\vec{v}, \vec{v} - \vec{w}] + B[\vec{v} - \vec{w}, \vec{w}]. \tag{3.18}$$

For $0 \leqslant t \leqslant T$, using (3.18) and Lemma 1, we have

$$\begin{aligned}
\|(S\vec{v} - S\vec{w})(t)\|_p \leqslant\; & \int_0^t \|e^{-(t-s)A} B[\vec{v}(s), (\vec{v} - \vec{w})(s)]\|_p \, ds \\
& + \int_0^t \|e^{-(t-s)A} B[(\vec{v} - \vec{w})(s), \vec{w}(s)]\|_p \, ds \\
\leqslant\; & C \int_0^t \frac{1}{(t-s)^{(\beta+1)/2}} \|(\vec{v} - \vec{w})(s)\|_p (\|\vec{v}(s)\|_p + \|\vec{w}(s)\|_p) \, ds \\
\leqslant\; & \frac{8C}{1-\beta} MT^{(1-\beta)/2} \sup_{0 \leqslant t \leqslant T} \|(\vec{v} - \vec{w})(s)\|_p,
\end{aligned}$$

where in the last inequality, we used (3.6). $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Recall that $C([0,T]; V_p)$ is a Banach space with respect to the norm $\|\vec{u}(\cdot)\| = \sup_{0 \leqslant t \leqslant T} \|\vec{u}(t)\|_p$. With δ as in Theorem 1, set

$$\beta = \frac{3(p-1) + \eta}{p}, \quad \text{where } \eta = \frac{\delta(3-2p)^2}{2p + \delta(3-2p)}.$$

Note that $\beta = 0$ if $p = 1$ and for $1 < p < \frac{3}{2}$, β satisfies $\frac{3(p-1)}{p} < \beta < 1$. Moreover, $2/(1-\beta) = 2p/(3-2p) + \delta$ for $1 \leqslant p < 3/2$. By Lemmas 2 and 3, if $T < \frac{C}{M^{2p/(3-2p)+\delta}}$ for appropriate C, then the map S defined in (3.7) is a contractive map from E into E. Thus, by Banach fixed point theorem, there exists a $\vec{u}(\cdot)$ in $E \subset C([0,T]; V_p)$ satisfying (2.11).

From (2.11) and the definition of the operator A, it follows $\vec{u}(\cdot)$ satisfies

$$\vec{u}(k,t) = e^{-t(\frac{2\pi}{L})^2|k|^2}\vec{u}_0(k) + \int_0^t e^{-(t-s)(\frac{2\pi}{L})^2|k|^2}\vec{g}(k,s)\,ds$$
$$- \int_0^t e^{-(t-s)(\frac{2\pi}{L})^2|k|^2} B[\vec{u}(s),\vec{u}(s)](k)ds, \quad (k \in \mathbb{Z}^3). \tag{3.19}$$

Note that for any vectors $\vec{w}, \vec{v} \in V_p$, $p \leqslant 2$ we have

$$|B[\vec{w},\vec{v}](k)| \leqslant |Q[\vec{w},\vec{v}](k)| \leqslant |k| \sum_{h\in\mathbb{Z}^3} |\vec{w}(h)||\vec{v}(k-h)|$$
$$\leqslant |k| \left(\sum_{h\in\mathbb{Z}^3} |\vec{w}(h)|^2\right)^{1/2} \left(\sum_{k\in\mathbb{Z}^3} |\vec{v}(k)|^2\right)^{1/2} \leqslant |k|\|\vec{w}\|_p\|\vec{v}\|_p, \tag{3.20}$$

where to obtain the first inequality in the second line of (3.20) we used Cauchy–Schwartz, while the last inequality in (3.20) follows from the fact that for $1 \leqslant p \leqslant q$ and $\vec{v} \in V_p$, $\|\vec{v}\|_q \leqslant \|\vec{v}\|_p$. Since $\vec{u}(\cdot)$ is in $C([0,T]; V_p)$, from (3.20) it immediately follows that for every $k \in \mathbb{Z}^3$, the map $s \to B[\vec{u}(s),\vec{u}(s)](k)$ is continuous on $[0,T]$. Thus we may differentiate (*a.e.*) under both the integral signs in (3.19) to conclude that $\vec{u}$ satisfies (2.10) *a.e.*

Remark 3. If $p = 1$, this is precisely the result obtained in [F].

We will now show that the mild solution $\vec{u}(\cdot)$ obtained in Theorem 1 is indeed a strong solution. We will need the following lemma.

Lemma 4. *Let $\vec{u} \in V_{\alpha,p}$. Then, for $\alpha \geqslant 0, \delta \geqslant 0, \eta > 0$ and $p \geqslant 1$, there exists a constant $C = C(\alpha,\delta,p)$ depending only on α, δ and p such that*

$$\|A^\delta e^{-\eta A}\vec{u}\|_{\alpha,p} \leqslant C\frac{1}{\eta^\delta}\|\vec{u}\|_{\alpha,p}. \tag{3.21}$$

Proof. The proof is a straightforward calculation as shown below.

$$\|A^\delta e^{-\eta A}\vec{u}\|_{\alpha,p}^p = \sum_k |k|^{\alpha p}\left(\frac{2\pi}{L}|k|\right)^{2\delta p} e^{-\eta p(\frac{2\pi}{L})^2|k|^2}|u(k)|^p$$
$$\leqslant C\frac{1}{\eta^{\delta p}}\sum_k |k|^{\alpha p}|u(k)|^p$$
$$= C\frac{1}{\eta^{\delta p}}\|\vec{u}\|_{\alpha,p}^p,$$

where to obtain the inequality above we used (3.8). □

We now proceed to prove Theorem 2.

Proof of Theorem 2. Using (2.9) and (3.21), for all $\alpha \geqslant 0$ we have

$$\sup_{0\leqslant t\leqslant T} t^{\alpha/2}\|e^{-tA}\vec{u}_0\|_{\alpha,p} = \left(\frac{2\pi}{L}\right)^{-\alpha} \sup_{0\leqslant t\leqslant T} t^{\alpha/2}\|A^{\alpha/2}e^{-tA}\vec{u}_0\|_p \leqslant C\|\vec{u}_0\|_p. \tag{3.22}$$

Furthermore, once again using (2.9) and (3.21), for $0\leqslant\alpha\leqslant 1$, we have

$$\begin{aligned}
\left\|\int_0^t e^{-(t-s)A}\vec{g}(s)\,ds\right\|_{\alpha,p} &= \left(\frac{2\pi}{L}\right)^{-\alpha}\left\|A^{\alpha/2}\int_0^t e^{-(t-s)A}\vec{g}(s)\,ds\right\|_p \\
&\leqslant \left(\frac{2\pi}{L}\right)^{-\alpha}\int_0^t \|A^{\alpha/2}e^{-(t-s)A}\vec{g}(s)\|_p\,ds \\
&\leqslant \int_0^t \frac{1}{(t-s)^{\alpha/2}}\|\vec{g}(s)\|_p\,ds \\
&\leqslant CT^{(1-\alpha/2)}\sup_{0\leqslant t\leqslant T}\|\vec{g}(t)\|_p < \infty.
\end{aligned}$$

To obtain the inequalities in the last line above, we used (3.21) and the fact that for $0\leqslant\alpha\leqslant 1$, $\int_0^T \frac{1}{\eta^{\alpha/2}}\,d\eta = \frac{2}{2-\alpha}T^{(1-\alpha/2)}$.

In view of the above estimates, for all $0\leqslant\alpha\leqslant 1$, we have

$$\sup_{0\leqslant t\leqslant T} t^{\frac{\alpha}{2}}\|\vec{G}(t)\|_{\alpha,p} < \infty. \tag{3.23}$$

We will now bootstrap regularity from a lower order bound. Assume (3.4) holds for some $0\leqslant\alpha_0 < 1$. Recall that by our assumption that $p < \frac{3}{2}$, and thus, $\frac{3(p-1)}{p} < 1$. Set $\delta = \frac{1-\beta}{4}$, where

$$\beta = 0 \text{ for } p = 1 \quad \text{and} \quad \beta = \frac{4p-3}{2p} \quad \text{for } 1 < p < 3/2. \tag{3.24}$$

Note that $(\beta+1)/2 + \delta < 1$ and moreover, $\frac{3(p-1)}{p} < \beta < 1$ for $1 < p < \frac{3}{2}$. Thus, $\delta > 0$ for $1\leqslant p < 3/2$. We have

$$\begin{aligned}
&\|A^\delta(S\vec{u} - G)(t)\|_{\alpha_0,p} \\
&\quad = \left\|A^\delta\int_0^t e^{-(t-s)A}B[\vec{u}(s),\vec{u}(s)]\,ds\right\|_{\alpha_0,p} \\
&\quad \leqslant \int_0^t \|A^\delta e^{-(t-s)A}B[\vec{u}(s),\vec{u}(s)]\|_{\alpha_0,p}\,ds
\end{aligned}$$

$$= \int_0^t \|A^\delta e^{-\frac{1}{2}(t-s)A} e^{-\frac{1}{2}(t-s)A} B[\vec{u}(s), \vec{u}(s)]\|_{\alpha_0,p}\, ds$$

$$\leqslant \int_0^t \frac{C}{(t-s)^\delta} \|e^{-\frac{1}{2}(t-s)A} B[\vec{u}(s), \vec{u}(s)]\|_{\alpha_0,p}\, ds \tag{3.25}$$

$$\leqslant \int_0^t \frac{C}{(t-s)^{\delta+(\beta+1)/2}} \|\vec{u}(s)\|^2_{\alpha_0,p}\, ds \tag{3.26}$$

$$\leqslant \left(\sup_{0\leqslant t\leqslant T} t^{\alpha_0} \|\vec{u}(t)\|^2_{\alpha_0,p} \right) \int_0^t \frac{C}{(t-s)^{\delta+(\beta+1)/2}} \frac{1}{s^{\alpha_0}} < \infty. \tag{3.27}$$

To obtain (3.25) above we used (3.21), and to obtain the inequality in (3.26), we used (3.9). In order to obtain the finiteness of the term in (3.27), we used induction assumption and the fact that $\alpha_0 < 1, (\beta+1)/2 + \delta < 1$ thus leading to the finiteness of the integral in that term. From (3.27) and (3.23), it immediately follows that (3.4) holds for all $0 \leqslant \alpha \leqslant 1$. This finishes the proof. □

4. Gevrey regularity

We will now proceed to obtain solution of NSE which is in Gevrey class. For $p \geqslant 1$ and $\gamma \geqslant 0$, the Gevrey class $X_{Gv(\gamma),p}$ is defined as

$$X_{Gv(\gamma),p} = \left\{ \vec{u} \in \mathcal{K} : \|\vec{u}\|_{Gv(\gamma),p} := \|e^{\gamma A^{1/2}} \vec{u}\|_p = \left(\sum_{k\in\mathbb{Z}^3} e^{p\gamma \frac{2\pi}{L}|k|} |u(k)|^p \right)^{1/p} < \infty \right\}.$$

Obviously, $X_{Gv(\gamma),p} \subset V_p$, and in fact, $\|\vec{u}\|_p \leqslant \|\vec{u}\|_{Gv(\gamma),p}$ for all $\gamma \geqslant 0$.

We will need a Young-type inequality for Gevrey norms.

Lemma 5. *Let $\vec{u}$, $\vec{v} \in X_{Gv(\gamma),p}$ and let $\vec{w} = (w(k))_{k\in\mathbb{Z}^3}$ be defined by $w(k) = \sum_h |\vec{u}(h)||\vec{v}(k-h)|$. Then, for $p < 2$,*

$$\|\vec{w}\|_{Gv(\gamma),r} = \left(\sum_k e^{\gamma r \left(\frac{2\pi}{L}\right)|k|} w(k)^r \right)^{1/r} \leqslant \|\vec{u}\|_{Gv(\gamma),p} \|\vec{v}\|_{Gv(\gamma),p}, \quad r = \frac{p}{2-p}. \tag{4.1}$$

Proof. Since $\vec{u}$ and $\vec{v}$ are in $X_{Gv(\gamma),p}$, the vectors $\vec{u}_1, \vec{v}_1$ defined by

$$\vec{u}_1(k) = e^{\gamma \frac{2\pi}{L}|k|} |\vec{u}(k)|, \quad \vec{v}_1(k) = e^{\gamma \frac{2\pi}{L}|k|} |\vec{v}(k)|, \quad (k \in \mathbb{Z}^3)$$

are in $\ell_p(\mathbb{Z}^3)$ and clearly,

$$\|\vec{u}_1\|_p = \|\vec{u}\|_{Gv(\gamma),p} \quad \text{and} \quad \|\vec{v}_1\|_p = \|\vec{v}\|_{Gv(\gamma),p}.$$

Consequently, by Young's inequality, the vector $\vec{w}_1 = \vec{u}_1 * \vec{v}_1$ is in $\ell_r(\mathbb{Z}^3)$, $r = \frac{p}{2-p}$ and

$$\|\vec{w}_1\|_r \leqslant \|\vec{u}_1\|_p \|\vec{v}_1\|_p \leqslant \|\vec{u}\|_{Gv(\gamma),p} \|\vec{v}\|_{Gv(\gamma),p}. \tag{4.2}$$

Using the inequality $|k| \leqslant |h| + |k-h|$ and noting that $\vec{w}_1(k) = (\vec{u}_1 * \vec{v}_1)(k) = \sum_h e^{\gamma\frac{2\pi}{L}|h|}|\vec{u}(h)|e^{\gamma\frac{2\pi}{L}|k-h|}|\vec{v}(k-h)|$ we have

$$\begin{aligned}\sum_k e^{\gamma r\frac{2\pi}{L}|k|} w(k)^r &= \sum_k \left[e^{\gamma\frac{2\pi}{L}|k|} \left(\sum_h |\vec{u}(h)||\vec{v}(k-h)| \right) \right]^r \\ &\leqslant \sum_k \left[\sum_h e^{\gamma\frac{2\pi}{L}|h|}|u(h)|e^{\gamma\frac{2\pi}{L}|k-h|}|v(k-h)| \right]^r = \sum_k \vec{w}_1(k)^r \\ &= \|\vec{w}_1\|_r^r \leqslant \|\vec{u}_1\|_p^r \|\vec{v}_1\|_p^r \leqslant \|\vec{u}\|^r_{Gv(\gamma),p} \|\vec{v}\|^r_{Gv(\gamma),p}.\end{aligned}$$

Thus we obtain (4.1). □

The following lemma, which plays the role of Lemma 1, provides an estimate of the nonlinear term in Gevrey norm.

Lemma 6. *Let $\vec{u}, \vec{v} \in X_{Gv(\gamma),p}$, $1 \leqslant p < 2$ and $\eta > 0$. Let $\beta = 0$ for $p = 1$ and $\beta > \frac{3(p-1)}{p}$ for $1 < p < \frac{3}{2}$. Then, there exists a constant C (which may depend on p, β) such that*

$$\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{Gv(\gamma),p} \leqslant C \frac{1}{\eta^{\frac{(\beta+1)}{2}}} \|\vec{u}\|_{Gv(\gamma),p} \|\vec{v}\|_{Gv(\gamma),p}. \tag{4.3}$$

Proof. Let $\vec{u}, \vec{v}$ be as in the lemma and let $\vec{w} = (w(k))_{k\in\mathbb{Z}^3}$ be defined by

$$w(k) = \sum_h |\vec{u}(h)||\vec{v}(k-h)|.$$

Then for $1 < p < 3/2$ we have

$$\|e^{-\eta A} B[\vec{u}, \vec{v}]\|^p_{Gv(\gamma),p} \leqslant \left(\frac{2\pi}{L}\right)^p \sum_k e^{-\eta p(\frac{2\pi}{L})^2|k|^2} e^{\gamma p\frac{2\pi}{L}|k|} \left| \sum_h (k\cdot\vec{u}(h))\vec{v}(k-h) \right|^p$$

$$
\begin{aligned}
&\leqslant \left(\frac{2\pi}{L}\right)^p \sum_k e^{-\eta p\left(\frac{2\pi}{L}\right)^2|k|^2} e^{\gamma p \frac{2\pi}{L}|k|}|k|^p|w(k)|^p \\
&= \left(\frac{2\pi}{L}\right)^p \sum_k |k|^{(\beta+1)p} e^{-\eta p(\frac{2\pi}{L})^2|k|^2} \frac{1}{|k|^{\beta p}} e^{\gamma p \frac{2\pi}{L}|k|}|k|^p|w(k)|^p \\
&\leqslant \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \sum_k \frac{1}{|k|^{\beta p}} e^{\gamma p \frac{2\pi}{L}|k|}|w(k)|^p \\
&\leqslant \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \left(\sum_k \frac{1}{|k|^{\frac{\beta p r}{r-p}}}\right)^{\frac{r-p}{r}} \left(\sum_k e^{\gamma r \frac{2\pi}{L}|k|}|w(k)|^r\right)^{\frac{p}{r}} \\
&= \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \|\vec{w}\|^p_{Gv(\gamma),r} \leqslant \frac{C}{\eta^{\frac{(\beta+1)p}{2}}} \|\vec{u}\|^p_{\gamma,p}\|\vec{v}\|^p_{\gamma,p},
\end{aligned} \tag{4.4}
$$

where to obtain the inequality in (4.4) we used (3.8). The two subsequent inequalities were obtained by first using Holder's inequality and then (4.1).

For $p = 1$, and $\beta = 0$, we proceed exactly as the derivation of the inequality in (4.4) to obtain

$$
\begin{aligned}
\|e^{-\eta A} B[\vec{u}, \vec{v}]\|_{Gv(\gamma),1} &\leqslant \frac{C}{\eta^{\frac{1}{2}}} \sum_k e^{\gamma \frac{2\pi}{L}|k|}|w(k)| \\
&= \frac{C}{\eta^{\frac{1}{2}}} \|\vec{w}\|_{Gv(\gamma),1} \leqslant \frac{C}{\eta^{\frac{1}{2}}} \|\vec{u}\|_{\gamma,1}\|\vec{v}\|_{\gamma,1},
\end{aligned}
$$

where the last inequality follows from (4.1). This finishes the proof of the lemma. $\square$

Throughout this section, we assume

$$0 \leqslant \mu \leqslant \frac{\pi}{L}.$$

Define the Banach space

$$
\begin{aligned}
C_G = \Bigg\{ &\vec{u}(\cdot) \in C([0,T]; V_p) : \|\vec{u}(\cdot)\|_{Gv} := \sup_{0 \leqslant t \leqslant T} \|e^{\mu t A^{1/2}} \vec{u}(t)\|_p \\
&= \sup_{0 \leqslant t \leqslant T} \|\vec{u}(t)\|_{Gv(\mu t),p} < \infty \Bigg\}.
\end{aligned} \tag{4.5}
$$

Let $\vec{G}(\cdot)$ be as in (3.1) and assume that

$$M := \|\vec{G}(\cdot)\|_{Gv} = \sup_{0 \leqslant t \leqslant T} \|e^{\mu t A^{1/2}} \vec{G}(t)\|_p < \infty. \tag{4.6}$$

Remark 4. Since $\mu \leqslant \frac{\pi}{L}$, the condition (4.6) is satisfied if

$$\vec{u}_0 \in V_p, \quad \text{and} \quad \int_0^T \|e^{\mu s A^{1/2}} \vec{g}(s)\|_p \, ds < \infty.$$

Let $E_1 \subset C_{Gv}$ be defined as

$$E_1 = \{\vec{v}(\cdot) \in C_{Gv} : \|\vec{v}(\cdot) - \vec{G}(\cdot)\|_{Gv} \leqslant M\}. \tag{4.7}$$

Lemma 7. *Assume that* $1 \leqslant p < \frac{3}{2}$ *and let* $\beta = 0$ *if* $p = 1$ *and* $\frac{3(p-1)}{p} < \beta < 1$ *for* $1 < p < \frac{3}{2}$. *For* $\vec{v}(\cdot) \in E_1$, *we have* $S\vec{v}$ *is in* C_{Gv} *and*

$$\|(S\vec{v} - G)(\cdot)\|_{Gv} \leqslant C(p, \beta) M^2 T^{(1-\beta)/2}. \tag{4.8}$$

Proof. Recall that for any $\gamma > 0$ we can write $\|\vec{u}\|_{Gv(\gamma),p} = \|e^{\gamma A^{1/2}} \vec{u}\|_p$. Moreover, since $\mu \leqslant \frac{\pi}{L}$

$$e^{-\beta[\frac{1}{2}A - \mu A^{1/2}]} \quad (\beta > 0) \text{ is a contraction on } V_p. \tag{4.9}$$

$$\begin{aligned}
\|(S\vec{v} - G)(t)\|_{Gv(\mu t),p} &= \|e^{\mu t A^{1/2}} (S\vec{v} - G)(t)\|_p \\
&\leqslant \int_0^t \|e^{\mu t A^{1/2}} e^{-(t-s)A} B[\vec{v}(s), \vec{v}(s)]\|_p \, ds \\
&\leqslant \int_0^t \|e^{\mu(t-s)A^{1/2}} e^{-\frac{(t-s)}{2}A} e^{-\frac{(t-s)}{2}A} e^{\mu s A^{1/2}} B[\vec{v}(s), \vec{v}(s)]\|_p \, ds \\
&\leqslant \int_0^t \|e^{-\frac{(t-s)}{2}A} e^{\mu s A^{1/2}} B[\vec{v}(s), \vec{v}(s)]\|_p \, ds \\
&= \int_0^t \|e^{-\frac{(t-s)}{2}} B[\vec{v}(s), \vec{v}(s)]\|_{Gv(\mu s),p} \, ds \\
&\leqslant C \int_0^t \frac{1}{(t-s)^{(\beta+1)/2}} \|\vec{v}(s)\|^2_{Gv(\mu s),p} \, ds \leqslant C M^2 T^{(1-\beta)/2},
\end{aligned}$$

where in the last inequality, we used Lemma 6 with $\eta = (t-s)/2$ and $\gamma = \mu s$ as well as the fact that $\sup_{0 \leqslant s \leqslant T} \|e^{\mu s A^{1/2}} \vec{v}(s)\|_p \leqslant 2M$. $\square$

We will next need a lemma which shows that S is a contraction on E with respect to the Gevrey norm on C_{Gv}.

Lemma 8. *For $\vec{v}, \vec{w} \in E_1$, and for p and β as in Lemma 7, we have*

$$\|(S\vec{v} - S\vec{w})(\cdot)\|_{Gv} \leqslant CMT^{(1-\beta)/2} \sup_{0\leqslant t\leqslant T} \|e^{\mu t A^{1/2}}(\vec{v} - \vec{w})(t)\|_p. \tag{4.10}$$

The proof of this lemma is analogous to the previous one and is omitted.

Theorem 3. *Let $1 \leqslant p < \frac{3}{2}$, $\mu < \frac{\pi}{L}$ and M be as in (4.6). Let $\delta_p = \delta > 0$ if $1 < p < \frac{3}{2}$; and $\delta_1 = \delta = 0$ for $p = 1$. Then, for an adequate constant $C = C(p, \delta)$ and $T < \frac{C}{M^{2p/(3-2p)+\delta}}$, there exists $\vec{u}(\cdot)$ in C_{Gv} with $\|\vec{u}(\cdot)\|_{Gv} = \sup_{0\leqslant t\leqslant T} \|e^{\mu s A^{1/2}}\vec{u}(s)\| < 2M$ which moreover satisfies*

$$\frac{d\vec{u}(t)}{dt} = -A\vec{u}(t) + \vec{g}(t) - B[\vec{u}(t), \vec{u}(t)], \quad \vec{u}(0) = \vec{u}_0 \quad a.e.\ 0\leqslant t\leqslant T. \tag{4.11}$$

The constant C depends on p only as $p \to \frac{3}{2}$ and on δ only as $\delta \to 0$.

Proof. For $1 \leqslant p < \frac{3}{2}$, set

$$\beta = \frac{3(p-1)+\eta}{p}, \quad \text{where } \eta = \frac{\delta(3-2p)^2}{2p + \delta(3-2p)}.$$

Note that $2/(1-\beta) = 2p/(3-2p)+\delta$ and $\beta = 0$ for $p = 1$. Moreover, for $1 < p < \frac{3}{2}$, β satisfies $\frac{3(p-1)}{p} < \beta < 1$. By Lemmas 7 and 8, if $T < \frac{C}{M^{2p/(3-2p)+\delta}}$ then for appropriate C, the map S defined in (3.7) is a contractive map from E_1 into E_1. Thus, by Banach fixed point theorem, there exists a $\vec{u}$ in $E_1 \subset C_{Gv}$ such that

$$\vec{u}(t) = e^{-tA}\vec{u}_0 + \int_0^t e^{-(t-s)A}\vec{g}(s)\,ds - \int_0^t e^{-(t-s)A}B[\vec{u}(s), \vec{u}(s)]\,ds.$$

The conclusion of the theorem now follows quite easily as in Theorem 1. $\square$

Acknowledgments

The author thanks Professor C. Foias for suggesting the problem and for subsequent discussions. Thanks are also due to Constantin Onica for carefully proof reading the manuscript and pointing out some errors. The work was supported in part by NSF Grant number DMS–0074460 and by the Junior Summer Research Fellowship (2003) from University of North Carolina at Charlotte.

References

[F] C. Foias, Navier–Stokes Equations, Texas A & M Lecture Notes, 2002.

[FK] H. Fujita, T. Kato, On the Navier–Stokes initial value problem I, Arch. Rational Mech. Anal. 16 (1964) 269–315.

[FT] C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier–Stokes equations, J. Funct. Anal. 87 (1989) 359–369.

[GK] Z. Grujić, I. Kukavica, Space analyticity for the Navier–Stokes and related equations with initial data in L^p, J. Funct. Anal. 152 (1998) 447–466.

[GM] Y. Giga, T. Miyakawa, Solutions in L_r of the Navier–Stokes initial value problem, Arch. Rational Mech. Anal. 89 (1985) 267–281.

[K] T. Kato, Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, Math. Z. 187 (1984) 471–480.

[M] K. Masuda, On the analyticity and the unique continuation theorem for solutions of the Navier–Stokes equation, Proc. Japan Acad. 43 (1967) 827–832.

[T] R. Temam, Navier–Stokes Equations and Nonlinear Functional Analysis, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66, second ed., SIAM, Philadelphia, PA, 1995.